Exercises Coalgebra for Lecture 12

The exercises labeled with **(*)** are optional and more advanced.

1. Let $(X, \to)$ be a labelled transition system over a set of labels A. Define, by induction, a predicate which holds for every process $p \in X$ such that all paths eventually end up in a stopped state (i.e., with no more transitions). Give your answer both in terms of inference rules, and as the least fixed point of a monotone function on a complete lattice.

2. Consider the lattice $\mathcal{P}(\mathbb{Z}^* \cup \mathbb{Z}^\omega)$ of (finite) lists and (infinite) streams over the integers $\mathbb{Z}$. We write $i : \sigma$ for concatenation of a list or stream σ and an $i \in \mathbb{Z}$. Consider the following inference rules.

$$\frac{\mathsf{pos}(\sigma) \quad i > 0}{\mathsf{pos}(i : \sigma)} \qquad \frac{}{\mathsf{pos}(\varepsilon)}$$

   where ε is the empty list.

   (a) Rephrase these inference rules as a monotone function on the lattice $\mathcal{P}(\mathbb{Z}^* \cup \mathbb{Z}^\omega)$.

   (b) What is a pre-fixed point of your function? What is the least pre-fixed point?

   (c) What is a post-fixed point of your function? What is the greatest post-fixed point?

3. In the lecture, we revisited the standard proof principle for induction over natural numbers (prove a property P by showing $P(0)$ and for all n: $P(n) \to P(n+1)$), in terms of pre-fixed points in a lattice. It is customary to weaken the induction step as follows: if $P(i)$ for all $i \leq n$, then $P(n+1)$. Reformulate this in terms of pre-fixed points and the lattice $\mathcal{P}(\mathbb{N})$.

4. Compute $\mathsf{Rel}(B)(R)$ for $B\colon \mathsf{Set} \to \mathsf{Set}$ given by

   (a) $B(X) = 2 \times X^A$

   (b) $B(X) = \mathcal{P}(A \times X)$.

   Derive, from the second answer, a notion of bisimulation between transition systems, seen as coalgebras $f\colon X \to \mathcal{P}(A \times X)$.

5. (From the second homework assigment of 2016; solutions on the webpage.) Let $(X, \to)$ be a labelled transition system over A. Let $A^\infty = A^\omega \cup A^*$ be the set of streams and (finite) words over A. The empty word is denoted by ε. Consider the following rules, involving a relation $\downarrow \subseteq X \times A^\infty$.

$$\frac{}{x \downarrow \varepsilon} \qquad \frac{x \xrightarrow{a} y \quad y \downarrow w}{x \downarrow aw} \tag{1}$$

   (for all $a \in A$, $w \in A^\infty$).

Let $\mathsf{Rel}_{X,A^\infty}$ be the set of relations of the form $R \subseteq X \times A^\infty$, partially ordered by subset inclusion $\subseteq$. This partial order is a complete lattice.

Given $x \in X$ and $w \in A^\infty$, we say w is a *trace* of x if there is a path from x labelled by w, that is, a path $x_1 \xrightarrow{a_1} x_2 \xrightarrow{a_2} x_3 \xrightarrow{a_3} \dots$ such that $x_1 = x$ and $w = a_1 a_2 a_3 \dots$. If $w \in A^*$ then we call this a *finite* trace, if $w \in A^\omega$ we call it an *infinite* trace.

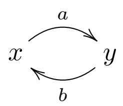
(a) Describe the least upper bound $\bigvee S$ and greatest lower bound $\bigwedge S$ of an arbitrary set $S \subseteq \mathsf{Rel}_{X,A^\infty}$, and give the top and bottom elements of the lattice (you don't have to give a proof).

(b) Formulate the rules (1) in terms of a function $b \colon \mathsf{Rel}_{X,A^\infty} \to \mathsf{Rel}_{X,A^\infty}$. Show that your function is monotone.

(c) What is a pre-fixed point of b? And what is the least fixed point? Give a concrete description, in terms of the transition system and elements of A^∞.

(d) What is a post-fixed point of b? And what is the greatest fixed point? Give a concrete description, in terms of the transition system and elements of A^∞.

(e) Use your answer to one of the previous questions to show that, in the transition system below, every finite trace of x is a prefix of the stream $ababab\dots$.



(f) Use your answer to one of the previous questions to show that, in the transition system above, the stream $ababab\dots$ is an infinite trace of x.

(g) Suppose that our transition system is finitely branching, meaning that for each $x \in X$, the set $\{y \mid x \xrightarrow{a} y \text{ for some } a\}$ is finite. Consider the relation $\| \subseteq X \times A^\infty$, given by: $x \| w$ iff there are infinitely many prefixes that are finite traces of x. Prove that for all $x \in X$ and $w \in A^\infty$: if $x \| w$, then w is an infinite trace of x.

6. **(*)** Complete the proof of the theorem stating that R is a Hermida-Jacobs bisimulation iff it is an Aczel-Mendler bisimulation.

7. **(*)** Consider the category Rel, where an object is a pair (R, X) such that $R \subseteq X \times X$, and an arrow $f \colon (R, X) \to (S, Y)$ is a function $f \colon X \to Y$ such that $(x, y) \in R \Rightarrow (f(x), f(y)) \in S$.

(a) Let $B \colon \mathsf{Set} \to \mathsf{Set}$ be a functor. Show that relation lifting $\mathsf{Rel}(B)$, as we've seen it in the lecture, extends to a functor $\mathsf{Rel}(B) \colon \mathsf{Rel} \to \mathsf{Rel}$.

(b) Show that a $\mathsf{Rel}(B)$-coalgebra is the same thing as a bisimulation.