Exercises Coalgebra, final week

Here are a few exercises that cover some of the main topics of the course: (final) coalgebras, homomorphisms, lattices and monads. Note that these do *not* at all cover the entire material of the course; they're meant for some extra practice with some of the harder topics, during the last exercise class.

1. 'Tis that time of the year again: time to put some (more) Christmas trees into the Mercator building. Of course, plain old finite trees would be boring: we're going to construct some infinite trees! So, we let B be a set of wonderful decorations for our tree.

   (a) We'll represent such trees by a set X of states together with some transitions: for each $x \in X$, *either* we're at a leaf, which we decorate with an element $b \in B$, or we are at a node; in that case, we get a pair (y, z) of states in X, modelling the left and right subtree respectively.
   Give a functor $F\colon \mathsf{Set} \to \mathsf{Set}$ whose coalgebras correspond to these systems.

   (b) Derive a concrete notion of homomorphisms between representations of Christmas trees, that is, F-coalgebras, from the coalgebraic notion.

   (c) Give a concrete notion of (coalgebraic) bisimulation for F-coalgebras.

   (d) Our Christmas trees will be of the form

   $$t\colon \{l, r\}^* \to B + 1\,,$$

   satisfying the property $t(w) \in B \to t(w) = t(wv)$ for all $w, v, \in \{l, r\}^*$.
   Show that the set of all such trees can be extended to a final F-coalgebra. Hint: after defining your coalgebra structure, first try and properly understand what homomorphisms into this coalgebra are.

   (e) Unfortunately, someone has put a number of tasteless diamonds $\Diamond \in B$ in the tree; we'd better replace these all by a nice box $\Box \in B$.
   Use that C is a final coalgebra to define a map $h\colon C \to C$ from the set C of Christmas trees to itself, which replaces every diamond by a box.

   (f) Several colleagues find the idea of infinite trees in Mercator a bit silly; help them out by giving the initial algebra of F (no proof needed).

   (g) Derive an induction principle for Christmas trees from your answer to the previous question.

   (h) Someone stole all the decorations; so that $B = \emptyset$. What are the initial algebra and final coalgebra in this case?

2. (Adapted from the second homework assigment of 2016; and the exercises for week 12) Let $(X, \to)$ be a labelled transition system over A. Let

$A^\infty = A^\omega \cup A^*$ be the set of streams and (finite) words over A. The empty word is denoted by ε. Consider the following rules, involving a relation $\downarrow \subseteq X \times A^\infty$.

$$\frac{}{x \downarrow \varepsilon} \qquad \frac{x \xrightarrow{a} y \quad y \downarrow w}{x \downarrow aw} \tag{1}$$

(for all $a \in A$, $w \in A^\infty$).

Let $\mathsf{Rel}_{X,A^\infty}$ be the set of relations of the form $R \subseteq X \times A^\infty$, partially ordered by subset inclusion $\subseteq$. This partial order is a complete lattice.

Given $x \in X$ and $w \in A^\infty$, we say w is a *trace* of x if there is a path from x labelled by w, that is, a path $x_1 \xrightarrow{a_1} x_2 \xrightarrow{a_2} x_3 \xrightarrow{a_3} \ldots$ such that $x_1 = x$ and $w = a_1 a_2 a_3 \ldots$. If $w \in A^*$ then we call this a *finite* trace, if $w \in A^\omega$ we call it an *infinite* trace.

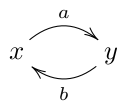
(a) Describe the least upper bound $\bigvee S$ and greatest lower bound $\bigwedge S$ of an arbitrary set $S \subseteq \mathsf{Rel}_{X,A^\infty}$, and give the top and bottom elements of the lattice.

(b) Formulate the rules (1) in terms of a function $b \colon \mathsf{Rel}_{X,A^\infty} \to \mathsf{Rel}_{X,A^\infty}$. Show that your function is monotone.

(c) Use the Kleene fixed point theorem (initial sequence $\bot, b(\bot), b(b(\bot), \ldots)$ to compute the least fixed point of b.

(d) Use the Kleene fixed point theorem (final sequence $\top, b(\top), b(b(\top), \ldots)$ to compute the greatest fixed point of b.

(e) Use your answer to one of the previous questions to show that, in the transition system below, every finite trace of x is a prefix of the stream $ababab\ldots$.



(f) Use your answer to one of the previous questions to show that, in the transition system above, the stream $ababab\ldots$ is an infinite trace of x.

(g) **(*)** Suppose that our transition system is finitely branching, meaning that for each $x \in X$, the set $\{y \mid x \xrightarrow{a} y \text{ for some } a\}$ is finite. Consider the relation $\| \subseteq X \times A^\infty$, given by: $x \| w$ iff there are infinitely many prefixes that are finite traces of x. Prove that for all $x \in X$ and $w \in A^\infty$: if $x \| w$, then w is an infinite trace of x.

3. As usual, the powerset functor is denoted by $\mathcal{P} \colon \mathsf{Set} \to \mathsf{Set}$, defined as usual by $\mathcal{P}(X) = \{U \mid U \subseteq X\}$ and on functions by taking direct image.

In an attempt to define a new monad, we try the following:

$$\eta_X \; : \; X \to \mathcal{P}(X), \quad x \mapsto \{x\} \qquad\qquad \mu_X \; : \; \mathcal{P}(\mathcal{P}(X)) \to \mathcal{P}(X), \quad S \mapsto \bigcap_{U \in S} U\,.$$

for any set X. Does this work: is it a monad?