# Exercises Coalgebra for Lecture 3

1. In the lecture, we have seen the *powerset functor*, which maps a set $X$ to the powerset $\mathcal{P}(X)$, and a function $f$ to $\mathcal{P}(f)$, $\mathcal{P}(f)(S) = \{f(x) \mid x \in S\}$.

   (a) Show that the *finite powerset* $\mathcal{P}_f$, defined on a set $X$ by

   $$\mathcal{P}_f(X) = \{U \subseteq X \mid U \text{ is finite}\}$$

   is also a functor (prove all the necessary properties).

   (b) (*) Consider the functor $F$ given on sets as the powerset: $F(X) = \mathcal{P}(X)$ but on a function $f\colon X \to Y$ by

   $$F(f)\colon \mathcal{P}(X) \to \mathcal{P}(Y)$$
   $$U \mapsto \{y \in Y \mid y \notin \{f(x) \mid x \in X \backslash U\}\}$$

   Is this a functor? Justify your answer (give a proof or a counterexample).

2. Let $F, G\colon \mathsf{Set} \to \mathsf{Set}$ be functors. Prove that $F + G$, given on a set $X$ by $(F + G)(X) = F(X) + G(X)$ is again a functor (how is it defined on functions?). Prove that the composition $F \circ G$, given on a set $X$ by $(F \circ G)(X) = F(G(X))$ is a functor.

3. Let $f\colon X \to Y$ be a function. Show that

   (a) $f$ is surjective if and only if there is a function $s\colon Y \to X$ such that $f \circ s = \mathsf{id}$.

   (b) Let $F\colon \mathsf{Set} \to \mathsf{Set}$ be a functor. Show that, if $f$ is surjective, then $F(f)$ is surjective as well.

   (c) How about injective functions? Can you prove something similar?

4. Let $A, B, C$ be sets, and consider the (exponent) polynomial functor given on sets by $F(X) = A + (B \times X)^C$. How is $F$ defined on functions, according to the definition of polynomial functors given in the lecture/section 2.2 of the book?

5. We have seen that stream systems over $A$ are coalgebras for the functor $F$ given by $F(X) = A \times X$.

   (a) What are coalgebras for the functor $G$ given by $G(X) = (A \times X) + 1$ (defined on a function $f$ as $G(f) = (\mathsf{id}_A \times f) + \mathsf{id}_1$), where $1 = \{*\}$ is a one-element set?

   (b) What is a homomorphism between such $G$-coalgebras? (Instantiate the general definition of coalgebra homomorphisms to the functor $G$, and spell out the details)

6. In the lecture, we saw that *pairing* $\langle f, g \rangle \colon Z \to X \times Y$ of functions $f \colon Z \to X$, $g \colon Z \to Y$ is the unique function making the following diagram commute:



where $\pi_1, \pi_2$ are the projections; that is, $\pi_1 \circ \langle f, g \rangle = f$ and $\pi_2 \circ \langle f, g \rangle = g$.

Give a similar picture for coproduct, using the *copairing* $[h, k] \colon X + Y \to Z$ of functions $h \colon X \to Z$, $k \colon Y \to Z$, using the injections $\kappa_1 \colon X \to X + Y$ and $\kappa_2 \colon Y \to X + Y$.

7. **(\*)** A *labelled transition system* over a set of labels $A$ consists of a set $X$ and a relation $\to \subseteq X \times A \times X$. We write $x \xrightarrow{a} y$ if $(x, a, y) \in \to$.

   (a) Show that labelled transition systems are coalgebras for the functor $F$ given by $F(X) = \mathcal{P}(A \times X)$.

   (b) What is a homomorphism from one labelled transition system to another? Instantiate the definition of coalgebra homomorphism for the above functor $F$, and spell out the details (in terms of transitions $x \xrightarrow{a} y$).

   (c) A labelled transition system $(X, \to)$ is *finitely branching* if, for each $x \in X$, the set $\{(a, y) \mid (x, a, y) \in \to\}$ is finite. Can you capture finitely branching labelled transition systems as coalgebras?

   (d) A labelled transition system $(X, \to)$ is *image-finite* if, for each $x \in X$ and $a \in A$, the set $\{y \mid (x, a, y) \in \to\}$ is finite. Can you capture image-finite labelled transition systems as coalgebras?