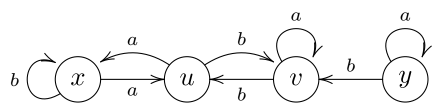
Coalgebra: homework assignment 2

December 6, 2018

If you have any questions, send me an email: `jrot@cs.ru.nl`. The deadline is December 21, 2018. You can hand it in by email, in my mailbox (first floor Mercator) or at an earlier class.

1. Give a bisimulation up to equivalence on the following automaton, which relates (x, y) and contains at most three pairs.



2. Show, using bisimulations up to congruence, that for all languages $L, K \in 2^{A^*}$, we have:

   (a) If $L \subseteq K$ then $LK^* \subseteq K^*$ (hint: use that $L \subseteq K$ iff $L + K = K$).

   (b) $(L^*K^*)^* = (L + K)^*$

   In both exercises, you may use basic algebraic properties such as idempotence ($L + L = L$), distributivity ($L(K + M) = LK + LM$) and so on.

3. Let A be a set. As usual, A^ω is the set of (infinite) streams, and A^* the set of (finite) lists over A. Given $\sigma \in A^\omega \cup A^*$, we let $|\sigma|$ be the number of elements of σ if σ is a list, and $|\sigma| = \infty$ if σ is a stream. Further, we denote by $\sigma(i)$ the i-th element of σ (if it exists), and for $|\sigma| > 0$ we denote by σ' the tail/derivative of σ.

   Consider the following inference rules for a predicate P on streams and lists:

$$\frac{|\sigma| < 2}{P(\sigma)} \qquad \frac{|\sigma| \geq 2 \qquad \sigma(0) \neq \sigma(1) \qquad P(\sigma')}{P(\sigma)}$$

   (a) Rephrase these inference rules as a monotone function on the complete lattice $L = \mathcal{P}(A^\omega \cup A^*)$ of sets of streams and lists, ordered by inclusion as usual. Don't forget to show that your function is indeed monotone.

   (b) What is a pre-fixed point of b? What is the least fixed point $\mathsf{lfp}(b)$? In both cases, give a concrete description in terms of lists and/or streams.

   (c) What is a post-fixed point of b? What is the greatest fixed point $\mathsf{gfp}(b)$? In both cases, give a concrete description in terms of lists and/or streams.

   (d) Take $A = 2$, so $L = \mathcal{P}(2^\omega \cup 2^*)$ is the set of binary streams and lists. Show that if $\sigma \in \mathsf{lfp}(b)$, then σ is of the form $(01)^i$, $(01)^i0$, $(10)^i$ or $(10)^i1$.

   (e) With $A = 2$ as in the previous exercise, show that $(01)^\omega = (0, 1, 0, 1, 0, 1, \ldots) \in \mathsf{gfp}(b)$.

4. In the lecture, we talked quite a bit about algebra and coalgebra. Excited about the combination, Jurriaan puts his favorite functors together with a natural transformation, defined as follows:

   - $B\colon \mathsf{Set} \to \mathsf{Set}$, $B(X) = \mathbb{N} \times X$,
   - $\mathcal{P}_f\colon \mathsf{Set} \to \mathsf{Set}$ the finite powerset functor (defined on functions by direct image),
   - a natural transformation $\lambda\colon \mathcal{P}_f B \Rightarrow B\mathcal{P}_f$, given on a component X by

$$\lambda_X\colon \mathcal{P}_f(\mathbb{N} \times X) \to \mathbb{N} \times \mathcal{P}_f(X)$$
$$S \mapsto \left(\sum_{(n,x)\in S} n, \{x \mid (n,x) \in S\} \right)$$

   He proudly shows this to Joshua. But Joshua frowns, and asks: is λ really a natural transformation? It's up to you to find out whether Joshua's worries are justified. Give either a proof or a counterexample for naturality of λ.

5. We define a functor $\mathcal{M}\colon \mathsf{Set} \to \mathsf{Set}$ by

$$\mathcal{M}(X) = \left\{ m\colon X \to \mathbb{N} \;\middle|\; |\{x \in X \mid m(x) \neq 0\}| \text{ is finite} \right\}$$

   on sets, and on functions by

$$\mathcal{M}(f\colon X \to Y)\colon \mathcal{M}(X) \to \mathcal{M}(Y)$$
$$\mathcal{M}(f)(m)(y) = \sum_{x \in f^{-1}(y)} m(x)\,.$$

   (a) Explain, in words, the difference between $\mathsf{List}(Y)$ and $\mathcal{M}(Y)$.
   (b) Define a non-trivial function from $\mathsf{List}(Y)$ to $\mathcal{M}(Y)$, using initiality of $\mathsf{List}(Y)$ with respect to $F_Y(X) := 1 + Y \times X$.
   (c) Define a monad structure on $\mathcal{M}$. You don't have to prove the equations or naturality (though doing so could result in bonus points!).

6. Let $f\colon X \to \mathcal{P}(A \times X)$ be a labelled transition system over a set of labels A. As usual, we let $\mathsf{Rel}_X = \mathcal{P}(X \times X)$ be the lattice of relations on X ordered by inclusion.

   (a) Define $b\colon \mathsf{Rel}_X \to \mathsf{Rel}_X$ such that R is a bisimulation iff $R \subseteq b(R)$.
   (b) Show that if $R \subseteq X \times X$ is an equivalence relation, then $b(R)$ is an equivalence relation as well.
   (c) Use the final sequence $X \times X \supseteq b(X \times X) \supseteq \ldots$ to compute the greatest bisimulation on the following transition system. Present the relations at each step in terms of partitions.