Radboud University

Properties of χ

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ESCADA

Introduction

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- as a map of n-periodic sequences;
- as a map of bi-infinite sequences;
- as a univariate polynomial;
- the function rule over other finite fields.

Outline

χ on *n*-periodic sequences

χ on bi-infinite sequences

Univariate forms of χ_n

Bounds on univariate forms of χ_n

Number of univariate representations of χ_n

Polynomial automorphisms

Known properties

- χ is shift-invariant,

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- Restricting χ to finite sequences of odd length, then it is bijective. [Daemen,1995]

χ on n-periodic sequences

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We can define χ on $\widehat{\mathbb{F}_2}$, or as χ_n on $\mathbb{F}_2^n$ (here indices modulo n).

χ_n and invertibility

Theorem (Daemen, 1995)

If n is odd, then χ_n is invertible.

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Theorem

If n is odd, then the order of χ_n is $2^{\lceil \lg(n) \rceil}$.

Visualisation

$$
\begin{array}{cccccccccccccccccccccccc|ccccccc}
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . & 1 \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . & . & . & 1 \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . & 1 & . & 1 & . & 1 \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . & . & . & . & . & . & . & 1 \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . & 1 & . & . & . & . & . & 1 & . & 1 \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . & . & . & 1 & . & . & . & 1 & . & . & . & 1 \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . & 1 & . & 1 & . & 1 & . & 1 & . & 1 & . & 1 & . & 1 \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 \\
. & . & . & . & . & . & . & . & . & . & . & . & 1 & . & 1 & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . & 1 \\
. & . & . & . & . & . & . & . & . & . & 1 & . & . & . & 1 & . & . & . & . & . & . & . & . & . & . & . & 1 & . & . & . & 1 \\
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. & . & 1 & . & . & . & 1 & . & . & . & 1 & . & . & . & 1 & . & . & . & 1 & . & . & . & 1 & . & . & . & 1 & . & . & . & 1 \\
1 & . & 1 & . & 1 & . & 1 & . & 1 & . & 1 & . & 1 & . & 1 & . & 1 & . & 1 & . & 1 & . & 1 & . & 1 & . & 1 & . & 1 & . & 1
\end{array}
$$

Example: χ_6

1 time:

Name: 1-cycle

Example: χ_6

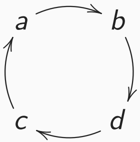
12 times:

$$a \rightleftarrows b$$

Name: 2-cycle

Example: χ_6

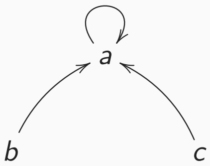
6 times:



Name: 4-cycle

Example: χ_6

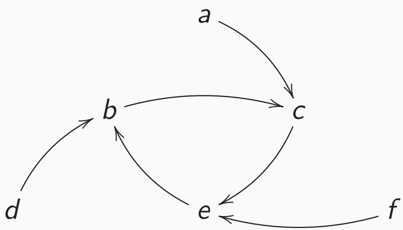
1 time:



Name: prong

Example: χ_6

2 times:



Name: spin

Example: χ_6 summary

shape	number	number of states
1-cycle	1	1
2-cycle	12	24
4-cycle	6	24
prong	1	3
spin	2	12

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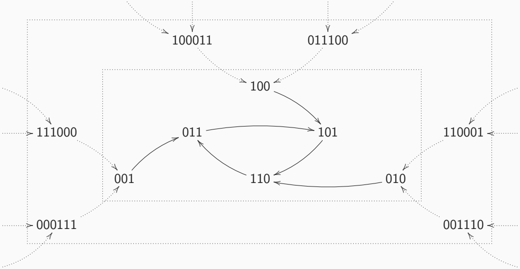
State diagram bigger example

We omit the zeroes in even positions, to obtain:



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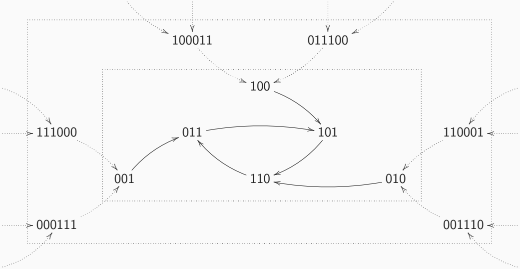

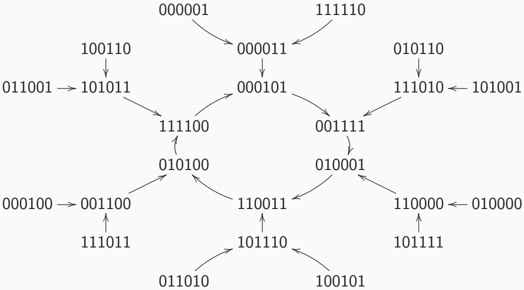

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  Reason:

$$\chi(x_0, 0, x_1, 0, \ldots, x_{n-1}, 0) = (x_0 + x_1, 0, x_1 + x_2, 0, \ldots, x_{n-1} + x_0, 0)$$

△

Proposition

Let $n = 2m$ with $m > 1$ an odd integer. Then the length of the cycle in a snowflake is a divisor of $2^o - 1$, where $o = \mathrm{ord}_{\mathbb{Z}/m\mathbb{Z}}(2)$.

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The length of the cycle in snowflakes of period n with $\frac{n}{2} = 2^k \cdot m$ with $m > 1$ odd, is 2^k times the length of the cycle in the snowflakes of period n with $\frac{n}{2} = m$.

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Theorem

Let $\sigma = (\sigma_0, \ldots, \sigma_{n-1})^*$ be a state in $S_{n,0}$. We have that σ is in the cycle if and only if $f_\sigma(X)$ has exactly 2^{k-1} divisors $X + 1$.

Snowflakes: Statements

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Furthermore, if $f_\sigma(X)$ has $2^{k-1} - \ell$ divisors $X + 1$, then $\chi^\ell(\sigma)$ is in the cycle.

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- However, there exists a $z \in \mathbb{F}_2^{2n}$ such that $\chi_{2n}(z) = y \| y$.
- We see that every element in Σ_n has a preimage in Σ_{2n}.

χ on bi-infinite sequences

Surjective on periodic states

Proposition

$\chi\colon \widehat{\mathbb{F}_2} \to \widehat{\mathbb{F}_2}$ *is surjective.*

Reason: Every element $x \in \widehat{\mathbb{F}_2}$ has a period n, so is in Σ_n. Then either it has a preimage in Σ_n, or it has a preimage in $\Sigma_{2n} \subset \widehat{\mathbb{F}_2}$. △

Can we give a concrete explanation whether $\chi\colon \mathbb{F}_2^{\mathbb{Z}} \to \mathbb{F}_2^{\mathbb{Z}}$ is surjective?

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$$\Delta = \cdots 000100101110100100010000100000100000010000000100000000100000000 \cdots$$

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Theorem (Tikhonov, 1935)

Let $(X, \mathcal{T})$ be a compact Hausdorff space and let $A \subset X$ be dense. Let $f : X \to X$ be a continuous map such that $f_{|A} \colon A \to A$ is surjective.

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Since, with $\chi(x) = y$, each y_i depends on only three bits of x, χ is continuous in the product topology, and thus χ is surjective.

Univariate forms of χ_n

Univariate expressions

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Univariate expressions

- Choosing an isomorphism (of vector spaces) from $\mathbb{F}_2^n$ to $\mathbb{F}_{2^n}$: χ_n as a univariate polynomial function: $\chi_n^u(t)$.
- In practice: interpolation on the inputs and outputs for χ_n to obtain $\chi_n^u(t)$.
- Different outcomes for $\chi_n^u(X)$ possible.

Univariate expressions

- Choosing an isomorphism (of vector spaces) from $\mathbb{F}_2^n$ to $\mathbb{F}_{2^n}$: χ_n as a univariate polynomial function: $\chi_n^u(t)$.
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- Example: $\chi_3^u(t) = t^6$. (With specific choice of basis $\{\alpha^3, \alpha^6, \alpha^5\}$ and $\mathbb{F}_2^3 \to \mathbb{F}_8,\ (a, b, c) \mapsto a\alpha^3 + b\alpha^6 + c\alpha^5$.)

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  $\mathrm{ord}(\chi_n) \geq 4$, but $\varphi(2^n - 1) = 2^n - 2$ has only one factor 2. $\triangle$

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- Less easy: χ_n is not a power function when $n > 3$.
  Done by investigating the differential probabilities for χ_n and power functions.

Definition (Differential probability [Biham, Shamir, 2009])

Let $f\colon G \to H$ be a map between finite groups G and H. Let $g \in G$ and $h \in H$ be arbitrary. Then we define the *differential probability of f at* (g, h) as

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input difference (columns); output difference (rows)

χ_3	000	001	010	011	100	101	110	111
000	1	-	-	-	-	-	-	-
001	-	1/4	-	1/4	-	1/4	-	1/4
010	-	-	1/4	1/4	-	-	1/4	1/4
011	-	1/4	1/4	-	-	1/4	1/4	-
100	-	-	-	-	1/4	1/4	1/4	1/4
101	-	1/4	-	1/4	1/4	-	1/4	-
110	-	-	1/4	1/4	1/4	1/4	-	-
111	-	1/4	1/4	-	1/4	-	-	1/4

Differential distribution for χ

Proposition (Differential probabilities for χ [Daemen,1995])

Let $n > 1$ be an arbitrary odd integer. Let $a \in \mathbb{F}_2^n$ be arbitrary. Then for any compatible $b \in \mathbb{F}_2^n$ we have $\mathrm{DP}_{\chi_n}(a, b) = 2^{-w(a)}$, where

$$w(a) = \begin{cases} n - 1 & \text{if } a = 1^n; \\ \mathrm{wt}(a) + r & \text{else,} \end{cases}$$

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- $a = 110^{n-2} \implies \mathrm{DP}_{\chi_n}(a, b) = \frac{1}{8}$;
- $a' = 10^{n-1} \implies \mathrm{DP}_{\chi_n}(a', b) = \frac{1}{4}$.

Invariant

Proposition (Differential probabilities under linear isomorphisms)

Let $G \cong H$ be isomorphic groups. Let $f : G \to G$ be a map and let $\hat{f} : H \to H$ be the map induced through the isomorphism φ. Then $\mathrm{DP}_{\hat{f}}(g, h) = \mathrm{DP}_f(\varphi^{-1}(g), \varphi^{-1}(h))$ for all $g, h \in H$.

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Let $0 \leq e \leq 2^n - 1$ and let $f = (\cdot)^e\colon \mathbb{F}_{2^n} \to \mathbb{F}_{2^n}$ be a power function. Then $\mathrm{DP}_f(a, b) = \mathrm{DP}_f(ya, y^e b)$ for all $y \in \mathbb{F}_{2^n}^*$.

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Proof.

Substitute $x := yy^{-1}x =: yx'$ in

$\mathrm{DP}_f(ya, y^e b) = \#\{x \in \mathbb{F}_{2^n} \mid x^e + (x + ya)^e = y^e b\}/2^n$. □

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Thus, we have that the rows of the DDT all have the same number of occurrences of $0, 2, 4, \ldots$.

χ is not a power function and corollary

Theorem

Let $n \neq 1, 3$ *be a positive integer. Then* χ_n^u *is not a power function.*

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Let $n \neq 1, 3$ be a positive integer. Then χ_n^u is not a power function.

Corollary

There is no function F_n that is extended affine equivalent to χ_n (i.e., $AF_nB + C = \chi_n$), such that F_n^u is a power function.

Bounds on univariate forms of χ_n

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n	3	5	7	9	11	13	15	17
$\max\deg(\chi_n^u)$	6	24	96	384	1,536	6,144	24,576	98,304
$2^n - 1$	7	31	127	511	2,047	8,191	32,767	131,071

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max. mon. in χ_n^u	6	15	28	45	66	91	120	153
2^n	8	32	128	512	2,048	8,192	32,768	131,072

Number of univariate representations of χ_n

Normal elements and normal bases

Definition (Normal basis)

Consider $\mathbb{F}_2 \subset \mathbb{F}_{2^n}$. Then $\beta \in \mathbb{F}_{2^n}$ is called a *normal element* of $\mathbb{F}_{2^n}$ over $\mathbb{F}_2$ if the set $\{\beta, \beta^2, \beta^{2^2}, \ldots, \beta^{2^{n-1}}\}$ is a linearly independent set.

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Let $F : \mathbb{F}_2^n \to \mathbb{F}_2^n$ be a shift-invariant map. Let β be a normal element of $\mathbb{F}_{2^n}$ and $\varphi_\beta : \mathbb{F}_2^n \to \mathbb{F}_{2^n}, \ (x_0, \ldots, x_{n-1}) \mapsto x_0\beta + \ldots + x_{n-1}\beta^{2^{n-1}}$.

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Let $n \geq 1$ be an integer. There exist precisely $\Phi_2(X^n - 1)/n$ normal elements in $\mathbb{F}_{2^n}$ (w.r.t. $\mathbb{F}_2$).

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  There are $\varphi(n)$ different orderings given a normal element.

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Let $f(X) = X^4 + X^3 + X + 1$, then $\Phi_2(f) = \Phi_2(X^2 + 1)\Phi_2(X^2 + X + 1) = 2 \cdot 3 = 6$.

Number of orderings of the normal basis

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$$\begin{array}{ccc}
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\tau^3 \Big\downarrow & & \\
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Thus: $\sigma = (1\ 3\ 4\ 2)$.

Polynomial automorphisms

Motivation

- For what finite fields $\mathbb{F}$ is a map $\xi_n \colon \mathbb{F}^n \to \mathbb{F}^n$, defined by

$$y_i = x_i + (x_{i+1} + 1)x_{i+2}$$

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- Related to the Jacobian conjecture!

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Theorem

Let $\mathbb{F}$ be an algebraically closed field and $F\colon \mathbb{F}^n \to \mathbb{F}^n$ an invertible polynomial function, then F is a polynomial automorphism.

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- The map $\xi_n \colon \overline{\mathbb{F}_2}^n \to \overline{\mathbb{F}_2}^n$, regarded as a polynomial map has a non-invertible Jacobian.

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Thank you for your attention!