Radboud University

Introduction to Finite Affine Geometry, with an application to Cryptography

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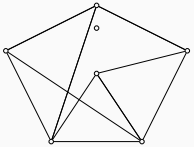
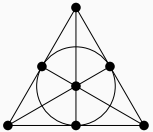




Euclidean Geometry example

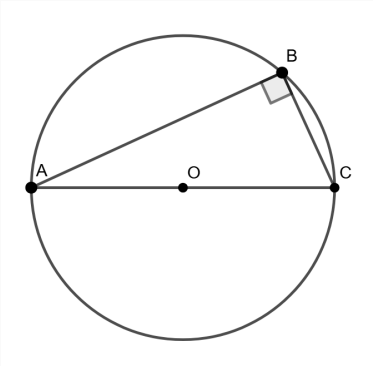


Figure: Thales' Theorem.

Euclidean Geometry

- (Mostly) planar geometry concerning points, lines and circles;
- Based on postulates:

(I): There is a line between any two points;
(II): Any line can be extended infinitely long;
(III): There is a circle for each center and radius;
(IV): All right angles are equal;
(V): (Parallel postulate) Given a line and a point not on the line, there is a line parallel to it through that point.[1]

[1] This is actually Playfair's axiom, which is equivalent.

- (Mostly) planar geometry concerning points and lines;
- Based on postulates:

(I): For any two distinct points, there is a line containing both;
(II): ~~Any line can be extended infinitely long;~~
(III): ~~There is a circle for each center and radius;~~
(IV): ~~All right angles are equal;~~
(V): (Playfair's axiom) Given a line ℓ and a point p not on ℓ, there exists a line ℓ' such that $\ell \cap \ell' = \emptyset$ and $p \in \ell'$.

Trivial examples

Definition (Finite Pre-Affine plane)

A finite pre-affine plane consists of a set of points P and a set of lines L such that it satisfies the postulates (I) and (V).

Example

Let $P = \emptyset$ and $L = \emptyset$. Then (P, L) satisfies (I) and (V) and thus is a finite pre-affine plane.

Example

Let $P = \{x_1 \ldots, x_n\}$ and $L = \{P\}$ (the line contains all points in P). Then (P, L) satisfies (I) and (V) and thus is a finite pre-affine plane.

Non-example and non-trivial example

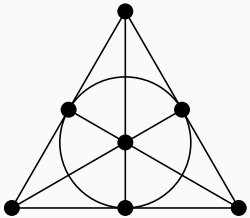
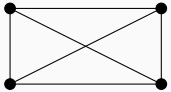


Figure: Fano plane.

The Fano plane has 7 points and 7 lines, where each line contains 3 points, each point lies on 3 lines and all lines intersect. It is actually an example of a projective plane.



This finite pre-affine plane has 4 points, 6 lines,and satisfies postulates (I) and (V).

Definition

A finite affine plane consists of a set of points P and a set of lines L such that it satisfies the postulates (I) and (V) and additionally that there is a set of four points in P such that no three of them are on the same line.

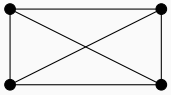


Figure: Affine plane of order 2.

Properties of finite affine planes

- All lines in a finite affine plane have the same number of points;
- All points in a finite affine plane are on the same number of lines;
- The number of points per line in a finite affine plane is called the order of the affine plane.
- Let $A = (P, L)$ be an affine plane of order k.
  - Every point is on $k + 1$ lines.
  - A has k^2 points.
  - For any line ℓ there are k lines that are parallel to ℓ.
  - A has $k^2 + k$ lines.

Construction (Coordinatization)

Let $\mathbb{F}$ be a field and consider $P := \mathbb{F} \times \mathbb{F}$. Furthermore, let $a, b, s \in \mathbb{F}$ and define $\ell_{s,b} := \{(x, y) \mid y = sx + b\}$ and $\ell_a := \{(a, y) \mid y \in \mathbb{F}\}$. Set $L = \{\ell_{s,b}\} \cup \{\ell_a\}_{a \in \mathbb{F}}$. Then (P, L) is an affine plane of order $\#\mathbb{F}$.

Example (Affine plane of order 2)

Let $P := \{(0,0), (0,1), (1,0), (1,1)\}$.

$\ell_0 := \{(0,0), (0,1)\}$; $\ell_1 := \{(1,0), (1,1)\}$;
$\ell_{0,0} := \{(0,0), (1,0)\}$; $\ell_{0,1} := \{(0,1), (1,1)\}$;
$\ell_{1,0} := \{(0,0), (1,1)\}$; $\ell_{1,1} := \{(0,1), (1,0)\}$.

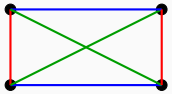


Figure: Affine plane of order 2.

Parallel lines

- Lines ℓ_a are parallel;
- Lines in $\{\ell_{s,b} \mid b \in \mathbb{F}\}$ are parallel for each slope s.
- We say that the lines ℓ_a have slope ∞.

Example

A finite affine plane of order 4. We take $\mathbb{F} := \mathbb{F}_2[X]/(X^2 + X + 1)$:

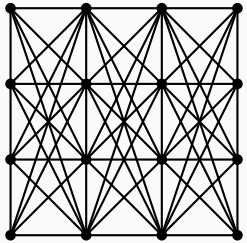


Figure: A finite affine plane of order 4.

Definition (Cluster)

Let X be a set of $n = s^2$ elements. A cluster C is a partition of X in s sets of equal size s.

Definition (MNO clusters)

Let C^1 and C^2 be two clusters of X, then they are maximally non-overlapping (MNO) if given any set $c \in C^1$ and any set $d \in C^2$, we have $|c \cap d| = 1$.

Definition (SMNO clusters)

Let $\mathfrak{C}$ be a collection of clusters. Then $\mathfrak{C}$ is called a collection of simultaneous MNO clusters if each pair of clusters in $\mathfrak{C}$ is MNO.

Maximum number of SMNO clusters

- Assume we have a finite affine plane $A = (P, L)$ of order s;
- Any set s of parallel lines is a cluster of P;
- Then any two clusters so obtained are MNO;
- Then we obtain $s + 1$ SMNO clusters.
- Known to be possible for all prime powers ($s = p^k$).

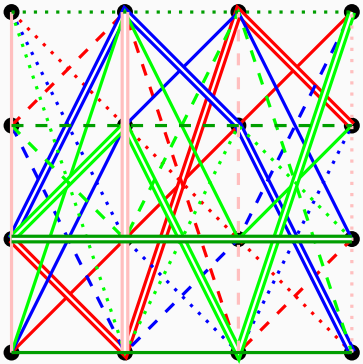


Figure: Example for $s = 4$.

To be continued...

- Masking is replaces gates like AND and XOR with gadgets;
- Gadgets are small circuits that operate on shared values and produce shared outputs;
- Gadgets typically require randomness;
- Use the clustering approach for making deterministic gadgets:
  - (uniform) AND-gadget;
  - Toffoli gadget;
  - higher-order gadgets to compute χ_5;
- Vahid will present about this two weeks from now!